

BRST operators for W algebras

A.P. Isaev^{a,*}, S.O. Krivonos^a and O.V. Ogievetsky^b

^a Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
Dubna, Moscow region 141980, Russia
E-mail: isaevap@theor.jinr.ru, krivonos@theor.jinr.ru

^b Center of Theoretical Physics[†], Luminy, 13288 Marseille, France
and P. N. Lebedev Physical Institute, Theoretical Department, Leninsky pr. 53, 117924
Moscow, Russia
E-mail: oleg@cpt.univ-mrs.fr

Abstract

The study of quantum Lie algebras motivates a use of non-canonical ghosts and anti-ghosts for non-linear algebras, like W -algebras. This leads, for the W_3 and $W_3^{(2)}$ algebras, to the BRST operator having the conventional cubic form.

*Corresponding author

[†]Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix-Marseille I, Aix-Marseille II et du Sud Toulon – Var; laboratoire affilié à la FRUMAM (FR 2291)

1 Introduction

The BRST symmetry was discovered in [1] and [2] in the context of gauge theories. The symmetry is generated by the BRST operator (BRST charge) Q whose square is zero, $Q^2 = 0$. It is used to describe the physical state space in constrained Hamiltonian (and Lagrangian) systems (see [3], [4] and references therein). The BRST operator Q is related to the homology of Lie algebras ([5], [6]).

The simplest BRST operator appears in theories with first class constraints χ_i , $i = 1, 2, \dots$; the constraints satisfy the Lie (super-)algebra relations

$$\chi_{i_1} \chi_{i_2} - (-1)^{(i_1)(i_2)} \chi_{i_2} \chi_{i_1} = C_{i_1 i_2}^{k_1} \chi_{k_1} . \quad (1.1)$$

Here $(k) \equiv 0, 1 \bmod 2$ is the parity of the generator χ_k , $(k) \equiv 0$ for a boson constraint χ_k and $(k) \equiv 1$ for a fermion constraint χ_k ; $C_{i_1 i_2}^{k_1}$ are the structure constants which satisfy the Jacobi identity together with

$$C_{mn}^k = 0 \quad \text{if } (k) + (n) + (m) \not\equiv 0 \quad (1.2)$$

and

$$C_{mn}^k = (-1)^{(m)(n)+1} C_{nm}^k . \quad (1.3)$$

To prepare the stage for general *quantum Lie algebras* (see the definition in Section 2) we rewrite (1.1) in the form

$$\chi_{i_1} \chi_{i_2} - \sigma_{i_1 i_2}^{k_1 k_2} \chi_{k_1} \chi_{k_2} = C_{i_1 i_2}^{k_1} \chi_{k_1} , \quad (1.4)$$

where

$$\sigma_{i_1 i_2}^{k_1 k_2} = (-1)^{(i_1)(i_2)} \delta_{i_2}^{k_1} \delta_{i_1}^{k_2} \quad (1.5)$$

is the super-permutation matrix. For the quantum Lie algebras, the operator σ is an invertible solution of the Yang–Baxter equation which is not necessarily the super-permutation. We shall see in Section 2 how to write the Jacobi identity and the conditions (1.2) and (1.3) in the general case.

The BRST operator for the algebra (1.1) is cubic in generators; it has the standard "two-term" form

$$Q = c^n \chi_n - \frac{1}{2} c^n c^m \phi_{mn}^{rs} C_{rs}^k b_k , \quad (1.6)$$

where the additional generators $\{c^n, b_m\}$ are the canonical (super) ghosts and anti-ghosts; we shall write the relations concerning the ghost—anti-ghost sector in the form adapted for the general quantum Lie algebras. The relations between ghosts—anti-ghosts and the generators χ_k are

$$b_m \chi_n = \phi_{mn}^{kl} \chi_k b_l , \quad \chi_m c^n = c^l \phi_{lm}^{kn} \chi_k , \quad (1.7)$$

where

$$\phi_{mn}^{kl} = (-1)^{(n)((m)+1)} \delta_n^k \delta_m^l . \quad (1.8)$$

The relations for ghosts—anti-ghosts are

$$\begin{aligned} b_{i_1} b_{i_2} &= -\tilde{\sigma}_{i_1 i_2}^{j_1 j_2} b_{j_1} b_{j_2} , \quad c^{k_2} c^{k_1} = -c^{j_2} c^{j_1} \tilde{\sigma}_{j_1 j_2}^{k_1 k_2} , \\ b_{i_2} c^{k_2} &= -c^{j_1} (\tilde{\sigma}^{-1})_{j_1 i_2}^{n_1 k_2} b_{n_1} + \delta_{i_2}^{k_2} , \end{aligned} \quad (1.9)$$

where

$$\tilde{\sigma}_{mn}^{kl} = (-1)^{(m)(n)+(m)+(n)} \delta_n^k \delta_m^l . \quad (1.10)$$

Note that

$$\tilde{\sigma}_{mn}^{kl} = \phi_{mn}^{pq} \sigma_{pq}^{rs} (\phi^{-1})_{rs}^{kl} \quad \text{or} \quad \tilde{\sigma} = \phi \sigma \phi^{-1} . \quad (1.11)$$

In (1.7) and (1.9) we have taken fermionic (respectively, bosonic) ghosts—anti-ghosts c^k and b_k for the bosonic (respectively, fermionic) constraints χ_k . There is however an alternative choice¹ of the ghost sector in the algebra: one can set the matrices ϕ and $\tilde{\sigma}$ to be the super-permutations (1.5),

$$\phi = \tilde{\sigma} = \sigma .$$

The relation (1.11) trivially holds for this choice as well.

The fact of having several choices for (1.7), (1.9) is a strong motivation for us to work with not necessarily canonical ghosts in the construction of the BRST operators for general quadratic algebras.

In this paper we discuss the form of the BRST operator for the gauge theories with constraints satisfying the quadratic relations (1.4) with the braid matrix σ which is not necessarily equal to the (super-)permutation matrix. We restrict our attention to the simplest (unitary) situation when $\sigma^2 = 1$. The BRST operator was investigated in the framework of 2D conformal theories for various non-linear algebras which include the Virasoro algebra (like W -type algebras), see, e.g., [8], [13], [14]. The interest in the construction of the BRST charge Q for quadratic algebras (1.4) has been recently renewed in [9, 10, 11] in the study of the higher spins in the AdS spaces (about quadratic algebras in this context see [12]). The main idea of our approach is to modify the ghost sector in the algebra in a way, compatible with the algebra of constraints (and not insisting on having the canonical ghosts, as it is done usually, see, e.g., [13] and [9]). More precisely, for the algebra defined by the quadratic relations (1.4) with a given braid matrix σ , we work with the quadratic ghost sector whose defining relations may contain the matrix σ as well.

The paper is organized as follows. In Section 2 we introduce non-canonical quadratic ghosts and anti-ghosts and generalize the construction of the BRST operator for Lie (super-)algebras to quantum Lie algebras with the unitary braid matrix σ . The data needed to define the ghost sector include a *twist pair* $\{\sigma, \phi\}$ of braid matrices. In Section 3 we consider two physical examples of quadratic algebras – the W_3 and $W_3^{(2)}$ algebras. As in section 2, we work with general quadratic ghost algebras. For the W_3 algebra,

¹With this choice, the BRST operator has the same form (1.6). The relation between two choices is discussed in [18], Section 6.

it turns out that the most general ghost sector includes two arbitrary parameters. We show that the BRST operator exists for arbitrary values of the parameters. The known BRST operators for the W_3 and $W_3^{(2)}$ algebras, based on the usual, canonical, ghosts and anti-ghosts, contain terms of degree higher than 3 in the generators. We find that for a certain choice of the values of the parameters (entering the definition of the ghost sector), the BRST operator has the simple conventional cubic form as in the case of the Lie (super-)algebras (1.6).

2 BRST operator for quadratic algebras

Consider an algebra \mathcal{A} with generators χ_i and defining quadratic-linear relations of the form

$$\chi_{i_1} \chi_{i_2} - \sigma_{i_1 i_2}^{k_1 k_2} \chi_{k_1} \chi_{k_2} = C_{i_1 i_2}^{k_1} \chi_{k_1} \quad (2.1)$$

or, in the concise notation ([15], [16]),

$$\chi_1 \rangle \chi_2 \rangle - \sigma_{12} \chi_1 \rangle \chi_2 \rangle = C_{12}^{\langle 1} \chi_1 \rangle . \quad (2.2)$$

The algebra \mathcal{A} is called *quantum Lie algebra* (QLA) if the structure constants σ_{kl}^{ij} and C_{ij}^k satisfy (for details see [17], [16] and references therein)

$$\sigma_{i_1 i_2}^{j_1 j_2} \sigma_{j_2 i_3}^{p_2 k_3} \sigma_{j_1 p_2}^{k_1 k_2} = \sigma_{i_2 i_3}^{j_2 j_3} \sigma_{i_1 p_2}^{k_1 p_2} \sigma_{p_2 j_3}^{k_2 k_3} , \quad (2.3)$$

$$C_{n_1 n_2}^{p_1} C_{p_1 n_3}^{k_4} = \sigma_{n_2 n_3}^{p_2 p_3} C_{n_1 p_2}^{k_1} C_{k_1 p_3}^{k_4} + C_{n_2 n_3}^{p_3} C_{n_1 p_3}^{k_4} , \quad (2.4)$$

$$C_{n_1 n_2}^{p_1} \sigma_{p_1 n_3}^{k_1 k_3} = \sigma_{n_2 n_3}^{p_2 p_3} \sigma_{n_1 p_2}^{k_1 j_2} C_{j_2 p_3}^{k_3} , \quad (2.5)$$

$$(\sigma_{n_2 n_3}^{j_2 p_3} C_{n_1 j_2}^{p_1} + \delta_{n_1}^{p_1} C_{n_2 n_3}^{p_3}) \sigma_{p_1 p_3}^{k_1 k_3} = \sigma_{n_1 n_2}^{j_2 p_3} (\sigma_{p_2 n_3}^{j_2 k_3} C_{p_1 j_2}^{k_1} + \delta_{p_1}^{k_1} C_{p_2 n_3}^{k_3}) , \quad (2.6)$$

$$\exists \ t_{jk}^i : C_{jk}^i = (\delta_j^l \delta_k^m - \sigma_{jk}^{lm}) t_{lm}^i . \quad (2.7)$$

Eq. (2.3) is the braid relation for the matrix σ_{ij}^{kl} and (2.4) is an analogue of the Jacobi identity. Eq. (2.5) reduces to (1.2) for Lie super-algebras. In the concise notation ([15], [16]), the identities (2.3) – (2.7) read

$$\sigma_{12} \sigma_{23} \sigma_{12} = \sigma_{23} \sigma_{12} \sigma_{23} , \quad (2.8)$$

$$C_{12}^{\langle 1} C_{13}^{\langle 4} = \sigma_{23} C_{12}^{\langle 1} C_{13}^{\langle 4} + C_{23}^{\langle 3} C_{13}^{\langle 4} , \quad (2.9)$$

$$C_{12}^{\langle 1} \sigma_{13} = \sigma_{23} \sigma_{12} C_{23}^{\langle 3} , \quad (2.10)$$

$$(\sigma_{23} C_{12}^{\langle 1} + C_{23}^{\langle 3}) \sigma_{13} = \sigma_{12} (\sigma_{23} C_{12}^{\langle 1} + C_{23}^{\langle 3}) , \quad (2.11)$$

$$C_{12}^{\langle 1} = (1 - \sigma_{12}) t_{12}^{\langle 1} . \quad (2.12)$$

Below we consider the simplest, unitary, braid matrices σ , that is,

$$\sigma_{nm}^{pj} \sigma_{pj}^{ki} = \delta_n^k \delta_m^i \quad \text{or} \quad \sigma^2 = 1 . \quad (2.13)$$

In this situation, eq. (2.11) follows from eq. (2.10) and eq. (2.12) is equivalent to (cf. (1.3) for Lie super-algebra case)

$$(1 + \sigma_{12})C_{12}^{(1)} = 0 . \quad (2.14)$$

Introduce the ghost generators c^i and b_i with defining quadratic relations

$$b_1 \rangle b_2 \rangle = -\tilde{\sigma}_{12} b_1 \rangle b_2 \rangle , \quad c^{(2)} c^{(1)} = -c^{(2)} c^{(1)} \tilde{\sigma}_{12} , \quad (2.15)$$

$$b_2 \rangle c^{(2)} = -c^{(1)} \tilde{\sigma}_{12}^{-1} b_1 \rangle + I_2 , \quad (2.16)$$

where I is the identity operator. We require that the matrix $\tilde{\sigma}_{kl}^{ij}$ satisfies the braid relation as well,

$$\tilde{\sigma}_{12} \tilde{\sigma}_{23} \tilde{\sigma}_{12} = \tilde{\sigma}_{23} \tilde{\sigma}_{12} \tilde{\sigma}_{23} , \quad (2.17)$$

which ensures that different reorderings of monomials $b_1 \rangle b_2 \rangle c^{(2)}$, $b_2 \rangle c^{(2)} c^{(1)}$ etc. give the same result. We note (and we shall not repeat it later) that the braid relation is stronger than the reordering requirement .

The appearance of another braid matrix $\tilde{\sigma}$ in (2.15), (2.16) follows the example of Lie super-algebras (see eqs. (1.7) and (1.9)) where the braid matrix σ in (2.2) is the super-permutation matrix (1.5) while $\tilde{\sigma}$ is defined in (1.10). Note that in particular one could identify the braid matrices $\tilde{\sigma}$ and σ . The possibility $\tilde{\sigma} = \sigma$ is also motivated by the differential calculi on quantum groups [7], [16]. Non-linear algebras with $\tilde{\sigma} = \sigma$ (for general non-unitary matrices σ) and the corresponding BRST operators were studied in [16], [18] and [19].

The smash product Ω of the QLA (2.2) and the ghost algebra (2.15), (2.16) is fixed by the cross-commutation relations

$$b_1 \rangle \chi_2 \rangle = \phi_{12} \chi_1 \rangle b_2 \rangle , \quad \chi_2 \rangle c^{(2)} = c^{(1)} \phi_{12} \chi_1 \rangle . \quad (2.18)$$

We require that the matrix ϕ_{kl}^{ij} satisfies relations

$$\tilde{\sigma}_{12} \phi_{23} \phi_{12} = \phi_{23} \phi_{12} \tilde{\sigma}_{23} , \quad \phi_{12} \phi_{23} \sigma_{12} = \sigma_{23} \phi_{12} \phi_{23} , \quad (2.19)$$

$$\phi_{12} \phi_{23} C_{12}^{(1)} \delta_3^{(2)} = C_{23}^{(2)} \phi_{12} , \quad (2.20)$$

which ensure that different reorderings of monomials $b_1 \rangle \chi_2 \rangle c^{(2)}$, $b_1 \rangle b_2 \rangle \chi_3 \rangle$, $b_1 \rangle \chi_2 \rangle \chi_3 \rangle$ etc. give the same result.

We now construct the BRST operator for the algebra Ω .

Proposition. *Let*

$$Q_{gh} := -\frac{1}{2} c^{(2)} c^{(1)} \phi_{12} C_{12}^{(1)} b_1 \rangle . \quad (2.21)$$

The operator (cf. (1.6))

$$Q = c^{(1)} \chi_1 \rangle + Q_{gh} \in \Omega \quad (2.22)$$

satisfies

$$Q^2 = 0 \quad (2.23)$$

if the matrix ϕ obeys (2.20) and defines the twist between the matrices σ and $\tilde{\sigma}$:

$$\tilde{\sigma}_{12} = \phi_{12} \sigma_{12} \phi_{12}^{-1}, \quad (2.24)$$

$$\sigma_{12} \phi_{23} \phi_{12} = \phi_{23} \phi_{12} \sigma_{23}, \quad \phi_{12} \phi_{23} \sigma_{12} = \sigma_{23} \phi_{12} \phi_{23}, \quad (2.25)$$

$$\phi_{12} \phi_{23} \phi_{12} = \phi_{23} \phi_{12} \phi_{23}. \quad (2.26)$$

Proof. We note that (2.17) and (2.19) follow from (or are contained in) (2.8), (2.24) and (2.25). In view of (2.13) and (2.24) we have $\tilde{\sigma}^2 = 1$.

1. First, $Q_{gh}^2 = 0$. Indeed,

$$\begin{aligned} 4 Q_{gh}^2 &= c^{\langle 4 \rangle} c^{\langle 3 \rangle} \phi_{34} C_{34}^{\langle 3 \rangle} b_3 \rangle c^{\langle 3 \rangle} c^{\langle 2 \rangle} \phi_{23} C_{23}^{\langle 2 \rangle} b_2 \rangle \\ &= c^{\langle 4 \rangle} c^{\langle 3 \rangle} \phi_{34} C_{34}^{\langle 3 \rangle} \left(c^{\langle 2 \rangle} c^{\langle 1 \rangle} \tilde{\sigma}_{23}^{-1} \tilde{\sigma}_{12}^{-1} b_1 \rangle + c^{\langle 2 \rangle} (1 - \tilde{\sigma}_{23}^{-1}) \right) \phi_{23} C_{23}^{\langle 2 \rangle} b_2 \rangle \\ &= c^{\langle 4 \rangle} \dots c^{\langle 1 \rangle} \phi_{34} C_{34}^{\langle 3 \rangle} \tilde{\sigma}_{23} \tilde{\sigma}_{12} \phi_{23} C_{23}^{\langle 2 \rangle} b_1 \rangle b_2 \rangle + c^{\langle 3 \rangle} c^{\langle 2 \rangle} c^{\langle 1 \rangle} \phi_{23} C_{23}^{\langle 2 \rangle} \phi_{12} (1 - \sigma_{12}) C_{12}^{\langle 1 \rangle} b_1 \rangle \\ &= c^{\langle 4 \rangle} \dots c^{\langle 1 \rangle} \phi_{1234} \sigma_{23} \sigma_{12} \sigma_{34} \sigma_{23} C_{34}^{\langle 3 \rangle} \phi_{23}^{-1} \phi_{12}^{-1} C_{23}^{\langle 2 \rangle} b_1 \rangle b_2 \rangle + 2 c^{\langle 3 \rangle} c^{\langle 2 \rangle} c^{\langle 1 \rangle} \phi_{123} C_{12}^{\langle 1 \rangle} \delta_3^{\langle 2 \rangle} C_{12}^{\langle 1 \rangle} b_1 \rangle \\ &= c^{\langle 4 \rangle} \dots c^{\langle 1 \rangle} \phi_{1234} C_{34}^{\langle 3 \rangle} C_{12}^{\langle 1 \rangle} \delta_3^{\langle 2 \rangle} \phi_{12}^{-1} b_1 \rangle b_2 \rangle + 2 c^{\langle 3 \rangle} c^{\langle 2 \rangle} c^{\langle 1 \rangle} \phi_{123} C_{12}^{\langle 1 \rangle} \delta_3^{\langle 2 \rangle} C_{12}^{\langle 1 \rangle} b_1 \rangle = 0. \end{aligned} \quad (2.27)$$

In the second equality we used (2.16); in the third equality we used $\tilde{\sigma}^2 = 1$ and relabeled the spaces in the second term; in the fourth equality we introduced, for $i < j$, the higher-rank matrices

$$\phi_{i,\dots,j} = (\phi_{i,i+1} \phi_{i+1,i+2} \dots \phi_{j-1,j}) (\phi_{i,i+1} \dots \phi_{j-2,j-1}) \dots (\phi_{i,i+1} \phi_{i+1,i+2}) \phi_{i,i+1},$$

which satisfy, by (2.24)–(2.26),

$$\phi_{i,\dots,j} \sigma_{i+k} = \tilde{\sigma}_{j-k-1} \phi_{i,\dots,j} \quad \text{for } k = 0, 1, \dots, j-i-1. \quad (2.28)$$

Here σ_l stands for $\sigma_{l,l+1}$ (same for $\tilde{\sigma}$). Then we took into account that, by (2.24)–(2.26), (2.10) and (2.20),

$$\begin{aligned} \phi_{34} C_{34}^{\langle 3 \rangle} \tilde{\sigma}_{23} \tilde{\sigma}_{12} \phi_{23} &= \phi_{34} C_{34}^{\langle 3 \rangle} \phi_{23} \sigma_{23} \phi_{23}^{-1} \tilde{\sigma}_{12} \phi_{23} = \phi_{234} C_{23}^{\langle 2 \rangle} \delta_4^{\langle 3 \rangle} \sigma_{23} \phi_{23}^{-1} \tilde{\sigma}_{12} \phi_{23} \\ &= \phi_{234} \sigma_{34} \sigma_{23} C_{34}^{\langle 3 \rangle} \phi_{23}^{-1} \tilde{\sigma}_{12} \phi_{23} = \phi_{234} \sigma_{34} \sigma_{23} C_{34}^{\langle 3 \rangle} \phi_{12} \tilde{\sigma}_{23} \phi_{12}^{-1} \\ &= \phi_{234} \sigma_{34} \sigma_{23} \phi_{12} \phi_{23} \phi_{34} C_{23}^{\langle 2 \rangle} \delta_4^{\langle 3 \rangle} \sigma_{23} \phi_{23}^{-1} \phi_{12}^{-1} = \phi_{1234} \sigma_{23} \sigma_{12} \sigma_{34} \sigma_{23} C_{34}^{\langle 3 \rangle} \phi_{23}^{-1} \phi_{12}^{-1} \end{aligned}$$

for the first term; we used (2.20) and (2.14) for the second term. The fifth equality in (2.27) uses (2.20) as well as (2.28) and

$$c^{\langle k} \dots c^{\langle 1} = c^{\langle k} \dots c^{\langle 1} A_k^{(\tilde{\sigma})} , \quad (2.29)$$

$$A_k^{(\tilde{\sigma})} \tilde{\sigma}_j = -A_k^{(\tilde{\sigma})} \quad \forall j < k , \quad (2.30)$$

where $A_k^{(\sigma)}$ are the anti-symmetrizing projectors, $(A_k^{(\sigma)})^2 = A_k^{(\sigma)}$, for the braid matrix σ constructed with the help of the recurrence relation

$$A_{k+1}^{(\sigma)} = \frac{1}{k!} (1 - \sigma_k + \sigma_{k-1} \sigma_k - \dots + (-1)^k \sigma_1 \dots \sigma_k) A_k^{(\sigma)} , \quad A_1^{(\sigma)} = 1 .$$

In the last, sixth, equality in (2.27) we used (2.29) as well as

$$b_{1\rangle} \dots b_{k\rangle} = A_k^{(\tilde{\sigma})} b_{1\rangle} \dots b_{k\rangle} ,$$

then (2.28) and then finally the identities (see [20] for details)

$$\begin{aligned} A_4^{(\sigma)} C_{34}^{\langle 3} C_{12}^{\langle 1} \delta_3^{\langle 2} (1 - \sigma_1) &= 0 , \\ A_3^{(\sigma)} C_{12}^{\langle 1} \delta_3^{\langle 2} C_{12}^{\langle 1} &= 0 . \end{aligned} \quad (2.31)$$

2. Since $Q_{gh}^2 = 0$, the verification of $Q^2 = 0$ reduces to

$$Q^2 \equiv (c^{\langle 2} \chi_2)^2 + [c^{\langle 1} \chi_1, Q_{gh}]_+ \stackrel{?}{=} 0 , \quad (2.32)$$

where $[\cdot, \cdot]_+$ is the anti-commutator and "?" denotes a statement to be verified. So we need to check only terms containing the generators χ_k :

$$c^{\langle 2} \chi_2 c^{\langle 2} \chi_2 - \frac{1}{2} \left(c^{\langle 1} \chi_1 c^{\langle 2} c^{\langle 1} \phi_{12} C_{12}^{\langle 1} b_1 \rangle + c^{\langle 2} c^{\langle 1} \phi_{12} C_{12}^{\langle 1} b_1 \rangle c^{\langle 1} \chi_1 \right) \stackrel{?}{=} 0 . \quad (2.33)$$

For the first term in l.h.s. of (2.33) we have

$$c^{\langle 2} \chi_2 c^{\langle 2} \chi_2 = c^{\langle 2} c^{\langle 1} \phi_{12} \chi_1 \chi_2 = \frac{1}{2} c^{\langle 2} c^{\langle 1} (1 - \tilde{\sigma}_1) \phi_{12} \chi_1 \chi_2 = \frac{1}{2} c^{\langle 2} c^{\langle 1} \phi_{12} C_{12}^{\langle 1} \chi_1 \rangle , \quad (2.34)$$

where we have used (2.18), (2.15), (2.24) and (2.2). For the second term in l.h.s. of (2.33) we have

$$\begin{aligned} & -\frac{1}{2} \left(c^{\langle 3} \chi_3 c^{\langle 3} c^{\langle 2} \phi_{23} C_{23}^{\langle 2} b_2 \rangle + c^{\langle 3} c^{\langle 2} \phi_{23} C_{23}^{\langle 2} b_2 \rangle c^{\langle 2} \chi_2 \right) \\ &= -\frac{1}{2} \left(c^{\langle 3} c^{\langle 2} c^{\langle 1} \phi_{23} \phi_{12} \phi_{23} C_{23}^{\langle 2} \phi_{12}^{-1} b_1 \rangle \chi_2 - \frac{1}{2} c^{\langle 3} c^{\langle 2} \phi_{23} C_{23}^{\langle 2} (-c^{\langle 1} \tilde{\sigma}_{12}^{-1} b_1 \rangle + I_2) \chi_2 \right) \\ &= -\frac{1}{2} c^{\langle 3} c^{\langle 2} c^{\langle 1} \left(\phi_{123} C_{23}^{\langle 2} \phi_{12}^{-1} - \phi_{23} C_{23}^{\langle 2} \tilde{\sigma}_{12} \right) b_1 \rangle \chi_2 - \frac{1}{2} c^{\langle 3} c^{\langle 2} \phi_{23} C_{23}^{\langle 2} \chi_2 \\ &= -\frac{1}{2} c^{\langle 3} c^{\langle 2} c^{\langle 1} \phi_{123} \left(C_{23}^{\langle 2} - \sigma_{23} \sigma_{12} C_{23}^{\langle 2} \right) \phi_{12}^{-1} b_1 \rangle \chi_2 - \frac{1}{2} c^{\langle 3} c^{\langle 2} \phi_{23} C_{23}^{\langle 2} \chi_2 . \end{aligned} \quad (2.35)$$

In the first equality we used (2.18) and (2.15); in the second equality we used $\tilde{\sigma}^2 = 1$; in the third equality we used (2.20), (2.26) and 2.10).

Substituting (2.34) and (2.35) into the expression for Q^2 , we obtain

$$Q^2 = -\frac{1}{2} c^{\langle 3} c^{\langle 2} c^{\langle 1} \phi_{123} (1 - \sigma_{23} \sigma_{12}) C_{23}^{\langle 2} \phi_{12}^{-1} b_1 \chi_2 \rangle = 0 .$$

Here we used (2.29) and took into account that

$$A_3^{(\tilde{\sigma})} \phi_{123} (1 - \sigma_{23} \sigma_{12}) = A_3^{(\tilde{\sigma})} (1 - \tilde{\sigma}_{12} \tilde{\sigma}_{23}) \phi_{123} = 0 \quad (2.36)$$

by (2.28) and (2.30). •

We conclude that the data for the construction of the algebra with ghosts include the twist pair $\{\sigma, \phi\}$, like in the study of the quantum matrix algebras and characteristic equations for them in [21].

As for the quantum matrix algebras, there are two natural possibilities to choose the twisting matrix ϕ for the general braid matrix σ . The first possibility is $\phi = \sigma^{\pm 1}$. The second possibility is to choose ϕ to be the super-permutation matrix (1.5). Then the ghosts simply (anti-)commute with the generators of the QLA (with this choice of ϕ , the twist equations (2.25) say only that the braid matrix σ should be even, i.e., $\sigma_{kl}^{ij} = 0$ if $(i) + (j) + (k) + (l) \neq 0$, see, e.g., [22]). For the purely even (bosonic) algebra \mathcal{A} , this second choice is the permutation matrix, $\phi_{nm}^{kl} = \delta_m^k \delta_n^l$, which leads to the direct tensor product of the QLA and the ghost algebra (2.16); the ghosts commute with the generators of the QLA,

$$b_i \chi_j = \chi_j b_i , \quad c^i \chi_j = \chi_j c^i . \quad (2.37)$$

The possibility (2.37) will be employed in the next Section for the algebras W_3 and $W_3^{(2)}$ (which have an infinite number of generators).

The bosonic ghost number operator G is well defined in the full algebra of constraints, ghosts and anti-ghosts,

$$[G, \chi_j] = 0 , \quad [G, b_j] = -b_j , \quad [G, c^j] = c^j \quad \text{for all } j \quad (2.38)$$

($[x, y] := xy - yx$ is the usual commutator; the BRST operator Q satisfies $[G, Q] = Q$), so one can induce representations from vacuum vectors and build Fock spaces in the usual manner.

For a given braid matrix σ_{kl}^{ij} (with $\sigma^2 = 1$), structure constants C_{jk}^i and a twisting matrix ϕ_{kl}^{ij} , the BRST operator Q of ghost number 1 (i.e. $[G, Q] = Q$) is defined uniquely by the requirement of having the conventional form $Q = c^i \chi_i + \text{ghost terms}$.

3 BRST operator for W_3 and $W_3^{(2)}$ algebras

We have seen in the previous Section that the BRST charge for quadratic algebras may be constructed with not necessarily canonical ghosts and anti-ghosts. The natural question

is whether, in the known examples, the structure of the BRST charge simplifies for a certain modification of the ghost—anti-ghost sector.

In this Section we consider two simplest quadratic algebras with an infinite number of generators, the W_3 and $W_3^{(2)}$ algebras. Their known BRST charges, based on the usual, canonical, ghosts and anti-ghosts, include terms of degree higher than 3 in the generators. We shall see that there is a freedom in the ghost—anti-ghost sector for these algebras. We demonstrate that for a certain choice of the ghost—anti-ghost sector, the BRST charges for these algebras can be written in the conventional, "degree three" form.

3.1 W_3 algebra

Our first example of a quadratic algebra with an infinite number of generators is the W_3 algebra discovered by A. Zamolodchikov in 1984 [23]. This algebra contains the Virasoro algebra, generated by the stress-tensor $T(z)$, and has, in addition, a spin 3 current $W(z)$. In terms of the Operator Product Expansion (OPE) the defining relations for the W_3 algebra have the form²

$$\begin{aligned} T(z_1)T(z_2) &\sim \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}}, & T(z_1)W(z_2) &\sim \frac{3W}{z_{12}^2} + \frac{W'}{z_{12}}, \\ W(z_1)W(z_2) &\sim \frac{c/3}{z_{12}^6} + \frac{2T}{z_{12}^4} + \frac{T'}{z_{12}^3} + \frac{a_1 T'' + a(TT)}{z_{12}^2} + \frac{a_2 T''' + \frac{a}{2}(TT)'}{z_{12}}, \end{aligned} \quad (3.1)$$

where $z_{12} = z_1 - z_2$ and

$$a = \frac{32}{22 + 5c}, \quad a_1 = \frac{3c - 6}{44 + 10c}, \quad a_2 = \frac{2}{9}a_1. \quad (3.2)$$

All currents in the r.h.s. of (3.1) are evaluated at the point z_2 and all products of currents are supposed to be normally ordered, $(AB)(z) = \frac{1}{2\pi i} \oint_z d\zeta \frac{A(\zeta)B(z)}{\zeta - z}$.

The main feature of the OPE's (3.1) is the appearance of the quadratic combinations of the currents T in the OPE of the W current with itself. These terms are absolutely necessary to fulfill the Jacobi identity for the currents T and W .

The BRST charge for the W_3 algebra (3.1) has been constructed by J. Thierry-Mieg in [8]. Explicitly the BRST charge reads

$$\tilde{Q} = \frac{1}{2i\pi} \oint dz Q(z), \quad (3.3)$$

²The general form of the OPE's is $A(z_1)B(z_2) = (\text{pole terms}) + (\text{non singular terms})$. In what follows we will explicitly write only pole terms in all OPE's.

where the BRST current $Q(z)$ has the following form³

$$\begin{aligned} Q = & (c_T T) + (c_W W) - (b_T c'_T c_T) - \frac{125}{1566} (b_T c'''_W c_W) - (c_T b_W c''_W) \\ & - \frac{25}{522} (b'_T c''_W c_W) + 2(c'_T b_W c_W) - \frac{8}{261} (T b_T c'_W c_W) . \end{aligned} \quad (3.4)$$

The ghosts—anti-ghosts currents (c_T, c_W, b_T, b_W) obey the standard OPE's:

$$b_T(z_1) c_T(z_2) \sim \frac{1}{z_{12}} , \quad b_W(z_1) c_W(z_2) \sim \frac{1}{z_{12}} \quad (3.5)$$

(all other OPE's for the ghosts—anti-ghosts are regular).

The BRST charge (3.3) squares to zero,

$$\tilde{Q}^2 = 0 , \quad (3.6)$$

if the central charge in the W_3 algebra (3.1) is equal to its critical value $c = 100$. With this central charge, the ghost modified current \tilde{T} ,

$$\tilde{T} = T + T_{gh} = T - (b'_T c_T) - 2(b_T c'_T) - 2(b'_W c_W) - 3(b_W c'_W) , \quad (3.7)$$

obeys the Virasoro algebra with the zero central charge

$$\tilde{T}(z_1) \tilde{T}(z_2) \sim \frac{2\tilde{T}}{z_{12}^2} + \frac{\tilde{T}'}{z_{12}} . \quad (3.8)$$

One checks that with respect to \tilde{T} the rest of the currents are primary, with the conformal dimensions

$$[W] = 3 , [c_T] = -1 , [b_T] = 2 , [c_W] = -2 , [b_W] = 3 , \quad (3.9)$$

as it should be.

The OPE's (3.1), (3.5) and the BRST current (3.4) are invariant under the following automorphism transformations

$$(T, c_T, b_T) \rightarrow (T, c_T, b_T) , \quad (W, c_W, b_W) \rightarrow (-W, -c_W, -b_W) . \quad (3.10)$$

Observe that the last term in the expression (3.4) for the BRST charge is unconventional, it has degree 4 in generators. Motivated by the discussions in the previous Section, we modify the ghost—anti-ghost algebra and study the resulting consequences for the BRST charge. The only restrictions we impose on the modified ghosts are: (i) the relations remain quadratic; (ii) the compatibility with the conformal weights (3.9); (iii)

³As usual, the BRST current is defined up to full derivatives which disappear after the integration in (3.3).

the invariance under the automorphism (3.10). One checks that the most general non-linear ghosts—anti-ghosts algebra, fulfilling these requirements, depends on two arbitrary parameters (g_1, g_2) . The corresponding OPE's read:

$$\begin{aligned}
\tilde{b}_T(z_1)\tilde{c}_T(z_2) &\sim \frac{1}{z_{12}} , \quad \tilde{b}_W(z_1)\tilde{c}_W(z_2) \sim \frac{1}{z_{12}} , \\
\tilde{c}_T(z_1)\tilde{b}_W(z_2) &\sim \frac{g_1(\tilde{b}_T\tilde{c}_W)}{z_{12}^2} + \frac{g_2(\tilde{b}_T\tilde{c}_W)' + g_1(\tilde{b}_T\tilde{c}_W')}{z_{12}} , \\
\tilde{c}_T(z_1)\tilde{c}_T(z_2) &\sim \frac{(g_1 + g_2)(\tilde{c}_W'\tilde{c}_W)}{z_{12}} , \quad \tilde{b}_W(z_1)\tilde{b}_W(z_2) \sim \frac{(g_1 - g_2)(\tilde{b}_T'\tilde{b}_T)}{z_{12}} .
\end{aligned} \tag{3.11}$$

The relations (3.11) define a quadratic algebra and the Jacobi identity is satisfied for arbitrary values of the parameters g_1 and g_2 .

Now this is a matter of calculations to check that the BRST current for the W_3 algebra with the new ghosts system $(\tilde{c}_{T,W}, \tilde{b}_{T,W})$ exists for arbitrary values of the parameters g_1 and g_2 . The BRST current has the form

$$\begin{aligned}
Q = & (\tilde{c}_T T) + (\tilde{c}_W W) - (\tilde{b}_T \tilde{c}_T' \tilde{c}_T) - \left[\frac{125}{1566} + \frac{17}{12}(g_1 + g_2) \right] (\tilde{b}_T \tilde{c}_W''' \tilde{c}_W) \\
& - (\tilde{c}_T \tilde{b}_W \tilde{c}_W'') - \left[\frac{25}{522} + \frac{5}{4}(g_1 + g_2) \right] (\tilde{b}_T' \tilde{c}_W'' \tilde{c}_W) + 2(\tilde{c}_T' \tilde{b}_W \tilde{c}_W) \\
& - \left[\frac{8}{261} + \frac{1}{2}(g_1 + g_2) \right] (T \tilde{b}_T \tilde{c}_W' \tilde{c}_W) - g_1(\tilde{b}_T' \tilde{b}_T \tilde{c}_T \tilde{c}_W' \tilde{c}_W) .
\end{aligned} \tag{3.12}$$

The corresponding BRST charge obeys the nilpotency condition (3.6) if the central charge in the algebra (3.1) equals $c = 100$ (independently of the values of the parameters g_1 and g_2), as it should be.

For general values of the parameters g_1 and g_2 , the BRST current contains unconventional terms – the third line in (3.12). The unconventional terms can be removed if we choose (and this choice is unique)

$$g_1 = 0 , \quad g_2 = -\frac{16}{261} . \tag{3.13}$$

Thus we see that the net effect of using the non-canonical ghost algebra is a freedom in the BRST charge. This freedom could be further fixed in a such way as to get the conventional form of BRST charge. Clearly, the standard, canonical, construction corresponds to the choice $g_1 = g_2 = 0$.

Finally, it is worth to note that for any values of the parameters g_1 and g_2 there is a non-linear transformation sending the old, canonical, ghosts—anti-ghosts currents (3.5) to the modified ghosts—anti-ghosts currents (3.11). It has the following form:

$$b_T = \tilde{b}_T , \quad c_T = \tilde{c}_T - \frac{g_1 + g_2}{2}(\tilde{b}_T \tilde{c}_W' \tilde{c}_W) , \quad b_W = \tilde{b}_W - \frac{g_1 - g_2}{2}(\tilde{b}_T' \tilde{b}_T \tilde{c}_W) , \quad c_W = \tilde{c}_W . \tag{3.14}$$

Although the transformation (3.14) is non-linear, it is clearly invertible.

Note that under this transformation (for $g_1 = 0$) the form of ghost stress-tensor

$$T_{gh} = -(\tilde{b}'_T \tilde{c}_T) - 2(\tilde{b}_T \tilde{c}'_T) - 2(\tilde{b}'_W \tilde{c}_W) - 3(\tilde{b}_W \tilde{c}'_W)$$

does not change.

3.2 $W_3^{(2)}$ algebra

Another example of the quadratic algebra with an infinite number of generators is provided by the so called $W_3^{(2)}$ algebra. In this subsection we demonstrate, without going into details, that the corresponding BRST current can be also brought to the conventional form by a proper modification of the ghost algebra.

The $W_3^{(2)}$ algebra [24], [25] is the bosonic analog of the well known $N = 2$ super Virasoro algebra. It contains four bosonic currents T, U, G^+, G^- with conformal dimensions

$$[T] = 2, [U] = 1, [G^+] = [G^-] = 3/2$$

which obey the following OPE's

$$\begin{aligned} T(z_1)T(z_2) &\sim \frac{c(7-9c)}{2(1+c)} \frac{1}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}}, \\ T(z_1)U(z_2) &\sim \frac{U}{z_{12}^2} + \frac{U'}{z_{12}}, \quad T(z_1)G^\pm(z_2) \sim \frac{\frac{3}{2}G^\pm}{z_{12}^2} + \frac{G^{\pm'}}{z_{12}}, \\ G^+(z_1)G^-(z_2) &\sim \left[\frac{2c-6c^2}{1+c} \right] \frac{1}{z_{12}^3} + \left[\frac{2-6c}{1+c} \right] \frac{U}{z_{12}^2} + \frac{2T - \frac{4}{1+c}(UU) + \frac{1-3c}{1+c}U'(z_2)}{z_{12}}, \\ U(z_1)G^\pm(z_2) &\sim \pm \frac{G^\pm}{z_{12}}, \quad U(z_1)U(z_2) \sim \frac{c}{z_{12}^2}. \end{aligned} \tag{3.15}$$

The BRST charge for this non-linear algebra has been constructed in [13], [26]. It obeys the nilpotency condition (3.6) for the critical value of the central charge $c = -2$, which corresponds to the Virasoro subalgebra central charge⁴ $c_{Vir} = 50$.

In a full analogy with the W_3 algebra, considered in the previous subsection, one checks that there is a conventional BRST current

$$\begin{aligned} Q &= (c_T T) + (\tilde{c}_U U) + (c^+ G^+) + (c^- G^-) + (\tilde{c}_U b^+ c^+) - (\tilde{c}_U b^- c^-) + \frac{1}{2}(c'_T b_U \tilde{c}_U) \\ &+ \frac{1}{2}(c_T b'_U \tilde{c}_U) - \frac{1}{2}(c_T b_U \tilde{c}'_U) + \frac{3}{4}(c'_T b^+ c^+) + \frac{1}{4}(c_T b^{+'} c^+) - \frac{3}{4}(c_T b^+ c^{+'}) \\ &+ \frac{3}{4}(c'_T b^- c^-) + \frac{1}{4}(c_T b^{-'} c^-) - \frac{3}{4}(c_T b^- c^{-'}) + 4(b_U c^+ c^{-'}) + 3(b'_U c^+ c^-) \\ &+ 2(b_U c^{+'} c^-) - (\tilde{b}_T c'_T c_T) - 2(\tilde{b}_T c^+ c^-) \end{aligned} \tag{3.16}$$

⁴The standard normalization for the Virasoro algebra reads $T(z_1)T(z_2) \sim \frac{c_{Vir}/2}{z_{12}^4} + \dots$

if the ghost—anti-ghost currents form a quadratically non-linear algebra

$$\begin{aligned}
\tilde{b}_T(z_1)c_T(z_2) &\sim \frac{1}{z_{12}} , \quad b_U(z_1)\tilde{c}_U(z_2) \sim \frac{1}{z_{12}} , \quad b^\pm(z_1)c^\pm(z_2) \sim \frac{1}{z_{12}} , \\
\tilde{b}_T(z_1)\tilde{c}_U(z_2) &\sim -2\frac{(c_T b_U)}{z_{12}^2} - 2\frac{2(c_T b'_U) + (c'_T b_U)}{z_{12}} , \\
\tilde{b}_T(z_1)\tilde{b}_T(z_2) &\sim -4\frac{(b'_U b_U)}{z_{12}} , \quad \tilde{c}_U(z_1)\tilde{c}_U(z_2) \sim -8\frac{(c^+ c^-)}{z_{12}} , \\
\tilde{c}_U(z_1)b^+(z_2) &\sim 4\frac{(b_U c^-)}{z_{12}} , \quad \tilde{c}_U(z_1)b^-(z_2) \sim -4\frac{(b_U c^+)}{z_{12}} .
\end{aligned} \tag{3.17}$$

As for the W_3 algebra, one can relate the ghost—anti-ghost currents $(c_T, \tilde{b}_T, \tilde{c}_U, b_U, c^\pm, b^\pm)$ obeying the OPE's (3.17) to the canonical ones $(c_T, b_T, c_U, b_U, c^\pm, b^\pm)$, obeying the standard OPE's

$$b_T(z_1)c_T(z_2) \sim \frac{1}{z_{12}} , \quad b_U(z_1)c_U(z_2) \sim \frac{1}{z_{12}} , \quad b^\pm(z_1)c^\pm(z_2) \sim \frac{1}{z_{12}} , \tag{3.18}$$

by a non-linear invertible transformation

$$\tilde{b}_T = b_T - 2(c_T b'_U b_U) , \quad \tilde{c}_U = c_U - 4(b_U c^+ c^-) \tag{3.19}$$

(only the currents b_T and c_U get transformed).

4 Conclusion

In this paper we have studied BRST operators for quadratic algebras. We have shown that the form of the BRST operator for the W_3 and $W_3^{(2)}$ algebras can be considerably simplified if one allows for a non-canonical quadratic ghost—anti-ghost sector instead of the canonical one. The possibility of having non-canonical ghosts—anti-ghosts is motivated by the BRST theory for Lie super-algebras (discussed in the Introduction) as well as by the Woronowicz theory [7] of differential calculi on quantum groups where the algebra of differential forms is deformed according to the algebra of non-commutative vector fields (the BRST operator in this case was constructed in [16], [18]). The vector fields in the differential calculus on quantum groups obey the reflection equation algebra (in dimension 2 this algebra is the q -Minkowski space, [29]) which is generated by entries of a matrix L with the relations

$$L_1 \hat{R}_{12} L_1 \hat{R}_{12} = \hat{R}_{12} L_1 \hat{R}_{12} L_1 . \tag{4.1}$$

Here \hat{R} is an arbitrary Yang–Baxter operator. If \hat{R}^2 is not proportional to one, the algebra (4.1) acquires linear terms upon the substitution $L \longrightarrow L + \alpha \text{Id}$, where $\alpha \neq 0$ is an arbitrary constant. The reflection equation algebra (with the shifted generators) is an example of a quantum Lie algebra; the ordering relations for copies of generators have the following form

$$L_1^{(A)} L_2^{(B)} = \hat{R}_{12} L_1^{(B)} L_2^{(A)} \hat{R}_{12}^{-1} + \alpha (\hat{R}_{12}^2 L_1^{(A)} \hat{R}_{12}^{-2} - L_1^{(A)}) \quad \text{for } A < B , \tag{4.2}$$

where $L_{\bar{2}} := \hat{R}_{12} L_1 \hat{R}_{12}^{-1}$; the upper index in $L^{(A)}$ labels the copy. The set (2.8)-(2.11) of the QLA identities for σ_{kl}^{ij} and C_{jk}^i is equivalent to the equality of two different reorderings of $\chi_1^{(A)} \chi_2^{(B)} \chi_3^{(C)}$ with $A < B < C$ (see [16] for details). To verify the QLA identities for the relations (4.2), it is convenient to reorder the combination $L_1^{(A)} L_{\bar{2}}^{(B)} L_{\bar{3}}^{(C)}$, where $L_{\bar{3}} := \hat{R}_{23} L_{\bar{2}} \hat{R}_{23}^{-1}$. The BRST operator for the reflection equation algebra was constructed in [30], [31].

We note that the phenomenon of the non-linear realization of the non-canonical ghosts via the canonical ones, observed in Section 3 for the W -algebras, is not universal and can not, in general, be realized for the quadratic algebras from Section 2, even if the number of generators is finite (cf. [32] where it is shown that the standard Drinfeld-Jimbo deformation of the Heisenberg relations leads to the algebra, which is isomorphic, up to a certain completion, to the non-deformed one).

Strictly speaking, the W -algebras from Section 3 are not defined in the form appropriate for the quantum Lie algebras; we don't know the braid matrix formulation for these algebras.

An intriguing question is how to write a non-linear ghost sector for a general non-linear algebra. For the W -algebras in particular, the appropriate non-linear ghost sector has to follow from the basic OPE's of the algebra we started from. An answer to this question could help to solve long standing problems of construction of the BRST charges for some infinitely generated super-algebras. In this respect the most interesting case is the structure of the BRST charge for the $N = 2$ super-symmetric W_3 algebra [27], [28]. The main problem here is the infinite Ansatz for the BRST charge (within the standard formulation) due to the presence of bosonic ghosts—anti-ghosts combinations with zero conformal weight and zero ghost number. We hope that the present approach will help in solving this problem, which probably will open a way to construct $N = 4$ super W_3 algebra along the lines presented in [33].

Acknowledgements

We are grateful to I. Buchbinder, R. Coquereaux, P. Lavrov, Ya. Pugai and A. Zamolodchikov for valuable discussions.

A part of this work was done while two of us (A.P.I. and S.O.K.) were visiting Marseille University and Centre de Physique Théorique (Luminy, Marseille). We thank Marseille University and Centre de Physique Théorique for the financial support. A.P.I. is also grateful to Centre International de Rencontres Mathématiques (Luminy, Marseille) for the kind hospitality and support. The work of A.P.I. was also partially supported by the grant RFBR-08-01-00392-a. The work of S.O.K. was partially supported by INTAS under contract 05-7928 and by grants RFBR-06-02-16684, 06-01-00627-a, DFG 436 Rus 113/669/03. The work of O.V.O. was supported by the ANR project GIMP No.ANR-05-BLAN-0029-01.

References

- [1] C. Becchi, A. Rouet and R. Stora, *Renormalization of the abelian Higgs-Kibble model*, Comm. Math. Phys. **42** (1975) 127.
- [2] I.V. Tyutin, *Gauge invariance in field theory and statistical physics in operator formalism*, preprint of Lebedev Physical Institute, No. 39 (1975).
- [3] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints*, Springer-Verlag (1990).
- [4] M. Henneaux and C. Teitelboim, *Quantization of gauge Systems*, Princeton Univ. Press (1992).
- [5] L. Bonora and P. Cotta-Ramusino, *Some remarks on BRS transformations, anomalies and the cohomology of the Lie algebra of the group of gauge transformation*, Comm. Math. Phys. **87** (1983) 589.
- [6] J. W. van Holten, *The BRST complex and the cohomology of compact Lie algebras*, Nucl. Phys. **B339** (1990) 158.
- [7] S. L. Woronowicz, *Differential calculus on compact matrix pseudogroups*, Comm. Math. Phys. **122** (1989) 125–170.
- [8] J. Thierry-Mieg, *BRS analysis of Zamolodchikov’s spin two and three current algebra*, Phys. Lett. B **197** (1987) 368.
- [9] I. L. Buchbinder and P. M. Lavrov, *Classical BRST charge for nonlinear algebras*; [arXiv: hep-th/0701243](#).
- [10] I. L. Buchbinder, A. Pashnev and M. Tsulaia, *Lagrangian formulation of the massless higher integer spin fields in the AdS background*, Phys. Lett. B **523** (2001) 338–346; [arXiv: hep-th/0109067](#).
- [11] I.L. Buchbidner, P.M. Lavrov, V.A. Krykhtin, *Gauge invariant Lagrangian formulation of higher spin massive bosonic field theory in AdS space*, Nucl. Phys. **B762** (2007) 344.
- [12] L. Brink, R. R. Metsaev and M. A. Vasiliev, *How massless are massless fields in AdS(d)*, Nucl. Phys. B **586** (2000) 183–205; [arXiv: hep-th/0005136](#).
- [13] K. Schoutens, A. Sevrin, P. van Nieuwenhuisen, *Quantum BRST Charge for Quadratically Nonlinear Lie algebras*, Comm. Math. Phys. **124** (1989) 87.
- [14] N. Ikeda, *Two-dimensional gravity and nonlinear gauge theory*, Annals Phys. **235** (1994) 435; [arXiv: hep-th/9312059](#).

- [15] L. D. Faddeev, N. Yu. Reshetikhin, and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990) 193–225 (Alg. Anal. **1** (1989) 178–206).
- [16] A. P. Isaev and O. V. Ogievetsky, *BRST operator for quantum Lie algebras and differential calculus on quantum groups*, Theor. Mat. Phys., **129**, No. 2 (2001) 1558–1572; [arXiv: math.QA/0106206](#).
- [17] D. Bernard, *A remark on quasitriangular quantum Lie algebras*, Phys. Lett. B **260** (1991) 389–393.
- [18] V. G. Gorbounov, A. P. Isaev and O. V. Ogievetsky, *BRST operator for quantum Lie algebras: relation to bar complex*, Theor. Math. Phys. **139** No. 1 (2004) 473–485; [arXiv: math.0711.4133 \[math.QA\]](#).
- [19] A. P. Isaev and O. V. Ogievetsky, *BRST operator for quantum Lie algebras: explicit formula*, Int. Journ. of Mod. Phys. A **19** Supplement (2004) 240.
- [20] A. P. Isaev and O. V. Ogievetsky, *BRST for QLA: Braid Combinatorics and Ghost Current*, to appear.
- [21] A. P. Isaev, O. V. Ogievetsky and P. N. Pyatov, *On quantum matrix algebras satisfying the Cayley-Hamilton-Newton identities*, J. Phys. A: Math. Gen. **32** (1999) L115–L121; [arXiv: math.QA/9809170](#).
- [22] A. P. Isaev, *Quantum groups and Yang-Baxter equations*, Sov. J. Part. Nucl. **26** (1995) 501; (extended version: preprint MPIM-Bonn, MPI-04 132 (2004), <http://www.mpim-bonn.mpg.de/Research/MPIM+Preprint+Series/>).
- [23] A. B. Zamolodchikov, *Infinite extra symmetries in two-dimensional conformal quantum field theory*, Theor. Math. Phys. **63** (1985) 1205.
- [24] A. M. Polyakov, *Gauge Transformations and Diffeomorphisms*, Int. J. Mod. Phys. A5 (1990) 833;
- [25] M. Bershadsky, *Conformal field theories via Hamiltonian reduction*, Commun. Math. Phys. **139** (1991) 71.
- [26] Z. Khviengia, E. Sezgin, *BRST operator for superconformal algebras with quadratic nonlinearity*, Phys. Lett. B326 (1994) 243; [arXiv: hep-th/9307043](#).
- [27] L. J. Romans, *The $N = 2$ super W_3 algebra*, Nucl. Phys. B369 (1992) 403;
- [28] E. Ivanov, S. Krivonos, *Superfield realizations of $N = 2$ super W_3* , Phys. Lett. B291 (1992) 63, Erratum-ibid. B301 (1993) 454.
- [29] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, *q -deformed Poincaré algebra*, Comm. Math. Phys. **150** (1992) 495–518.

- [30] A. P. Isaev and O. V. Ogievetsky, *BRST and anti-BRST Operators for Quantum Linear Algebra $U_q(gl(N))$* , Nucl. Phys. **B102** (2001), 306–311.
- [31] C. Burdik, A. P. Isaev and O. V. Ogievetsky, *Standard Complex for Quantum Lie Algebras*, Phys. Atomic Nuclei **64** (2001), no. 12, 2101–2104 (Yadernaya Fiz. 64 (2001), no. 12, 2191–2194); [arXiv: math.QA/0010060](#).
- [32] O. Ogievetsky, *Differential Operators on Quantum Spaces for $GL_q(n)$ and $SO_q(n)$* , Lett. Math. Phys. **24**, 245 (1992).
- [33] M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, *Extended $N=2$ superconformal structure of gravity and W gravity coupled to matter*, Nucl.Phys. B401 (1993) 304-347; [arXiv: hep-th/9211040](#).